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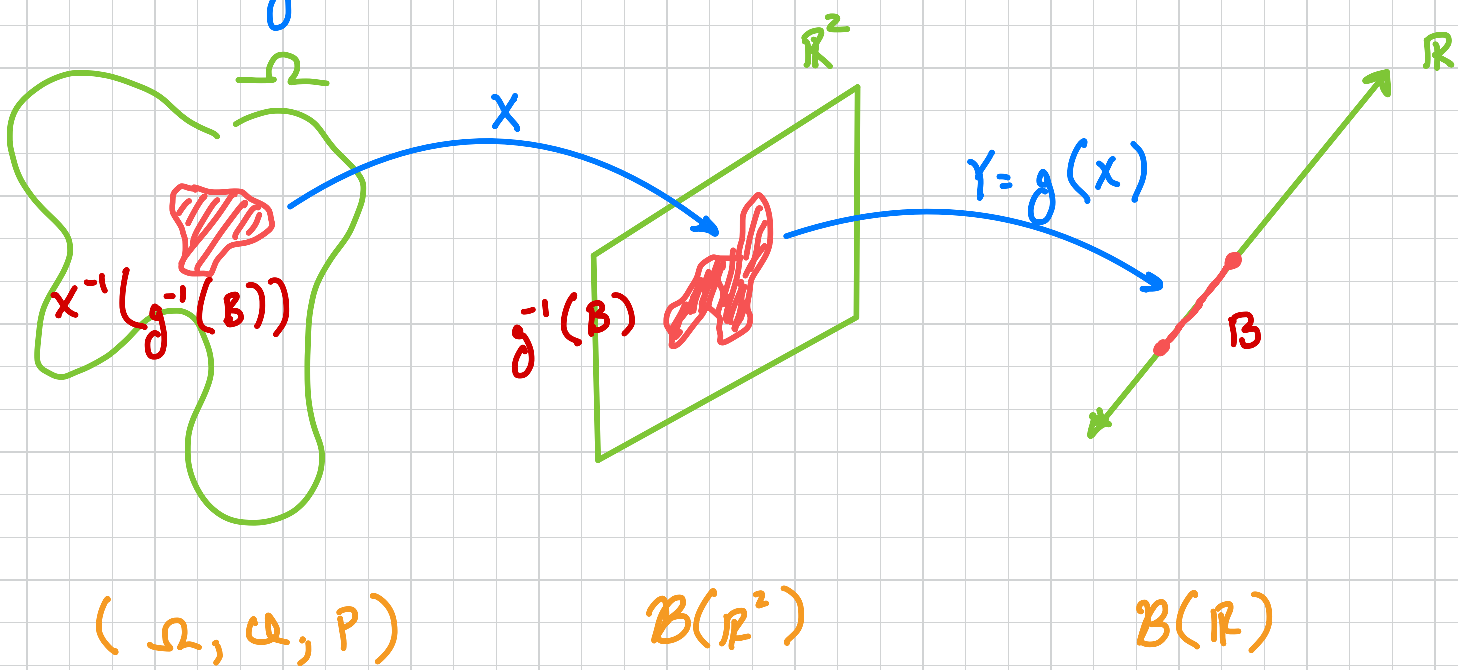
EE 364 - Spring 2026

Week 7
Supplemental Notes

Transformed Random Variables

$Y = g(X)$



$\therefore$ Work w/ distributions because of PAM (not densities, pdfs)
↳ CDF (or pmf)

Ex: discrete probability, 1-to-1

$Y = g(X)$ — g: 1-to-1, pmf: $p_X(x)$

$$p_Y(y) = P[Y = y] = P[g(X) = y] = P[\underbrace{g^{-1}(g(X))}_{g^{-1}\circ g = \text{identity}} = g^{-1}(y)]$$
$$= P[X = g^{-1}(y)] = p_X(g^{-1}(y)).$$

Ex: $X \sim \text{Geometric}(p)$ $\qquad$ $y = g(x) = x - k$ (1-to-1)

$\therefore\ x = g^{-1}(y) = y + k$

$$\therefore\ p_X(x) = \begin{cases} p \cdot (1-p)^{x-1} & x = 1, 2, \ldots \\ 0 & \text{else} \end{cases}$$

$$\therefore\ p_Y(y) = p_X(\underbrace{g^{-1}(y)}_{y+k}) = \begin{cases} p(1-p)^{(y+k)-1} & y+k = 1, 2, \ldots \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} p(1-p)^{y+k-1} & y = 1-k, 2-k, \ldots \\ 0 & \text{else} \end{cases}$$

"Simple" functions of random variables

Suppose X is a random variable

$g(x)$ is a function $\longrightarrow$ put $Y = g(X)$

$\therefore$ Y is a random variable

Q: $P[Y \in C] = P[g(X) \in C] \overset{?}{=} P[X \in B]$

e.g. $0 \le Y \le 2$

Q: what is equivalent B (w.r.t. X)

A: depends on X and g.

Ex: $Y = aX + b$ $\qquad$ Suppose $X \sim \text{CDF}: F_X(x)$

$$F_Y(y) = P[Y \le y] = P[aX + b \le y]$$

Case 1: $a > 0$, then $aX + b \le y \longleftrightarrow X \le \frac{y-b}{a}$

$$\therefore\ P[Y \le y] = P[aX + b \le y] = P\left[X \le \frac{y-b}{a}\right] = F_X\left(\frac{y-b}{a}\right)$$

Case 2: $a < 0$, then $aX + b \le y \longleftrightarrow X \ge \frac{y-b}{a}$

$$\therefore\ P[Y \le y] = P[aX + b \le y] = P\left[X \ge \frac{y-b}{a}\right] = 1 - F_X\left(\frac{y-b}{a}\right)$$

Q: how about pdf: f_Y ?

A: $\frac{dF_Y}{dy} = \frac{dF_Y}{dx} \cdot \frac{dx}{dy}$ w/ $x = \frac{y-b}{a}$

∴ Case 1: $a > 0$ $\quad f_Y(y) = \frac{d}{dy} F_X\left(\frac{y-b}{a}\right) = \frac{1}{a} \cdot f_X\left(\frac{y-b}{a}\right)$

Case 2: $a < 0$ $\quad f_Y(y) = \frac{d}{dy}\left(1 - F_X\left(\frac{y-b}{a}\right)\right)$

$$= \underbrace{-\frac{1}{a}}_{a<0 \;\therefore\; -\frac{1}{a} = \frac{1}{|a|}} \cdot f_X\left(\frac{y-b}{a}\right)$$

$$\therefore f_Y(y) = \frac{1}{|a|} \cdot f_Y\left(\frac{y-b}{a}\right)$$

Ex: $Z \sim N(0,1)$ $\quad f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

$X = 3Z + 2$ $\quad \therefore f_X(x) = \frac{1}{|3|} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{y-2}{3}\right)^2} = \frac{1}{\sqrt{2\pi}\,3} \cdot e^{-\frac{1}{2}\left(\frac{y-2}{3}\right)^2}$

$$\sim N(2, \underset{3^2}{9})$$

$Y = -3Z - 2$ $\quad \therefore f_Y(y) = \frac{1}{|-3|} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{y+2}{-3}\right)^2} = \frac{1}{\sqrt{2\pi}\,3} \cdot e^{-\frac{1}{2}\left(\frac{y-2}{3}\right)^2}$

$$\sim N(2, \underset{3^2}{9})$$

Thrm: Suppose random variable X, with pdf $f_X(x)$

Put $Y = g(X)$ then $f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{dx}{dy} \right|$ $\leftarrow \frac{dg^{-1}}{dy}$

g: 1-to-1 and differentiable.

Prf:

g 1-to-1 $\longrightarrow$ g^{-1} inverse exists

g differentiable $\longrightarrow$ continuous $\xrightarrow{\text{1-to-1}}$ g ① strictly-increasing or ② strictly-decreasing

① **Case 1:** g strictly increasing $\therefore$ g^{-1} strictly increasing.

$\therefore \frac{dg^{-1}}{dy} > 0$

$$F_Y(y) = P[Y \le y] = P[g(X) \le y] = P[g^{-1}(g(X)) \le g^{-1}(y)]$$
$$= P[X \le g^{-1}(y)] = F_X(g^{-1}(y))$$

$$\therefore f_Y(y) = \frac{dF_Y}{dy} = \frac{dF_Y}{dx} \cdot \frac{dx}{dy} = \frac{d}{dx}\left(F_X(g^{-1}(y))\right) \cdot \frac{dx}{dy}$$

w/ $x = g^{-1}(y)$
$$= f_X(g^{-1}(y)) \cdot \frac{dx}{dy}$$

② **Case 2:** g strictly decreasing $\therefore$ g^{-1} strictly decreasing

$\therefore \frac{dg^{-1}}{dy} < 0$

$$F_Y(y) = P[Y \le y] = P[g(X) \le y] = P[g^{-1}(g(X)) \ge g^{-1}(y)]$$
$$= P[X \ge g^{-1}(y)] = 1 - F_X(g^{-1}(y))$$

$$\therefore f_Y(y) = \frac{dF_Y}{dy} = \frac{dF_Y}{dx} \cdot \frac{dx}{dy} = \frac{d}{dx}\left(1 - F_X(g^{-1}(y))\right) \cdot \frac{dx}{dy}$$

w/ $x = g^{-1}(y)$
$$= f_X(g^{-1}(y)) \cdot \left(-\frac{dx}{dy}\right)$$

$-\frac{dx}{dy} = \left| \frac{dx}{dy} \right|$

$$\therefore f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{dx}{dy} \right|$$

Ex: Linear $Y = g(X) = aX + b$ w/ $a \neq 0$.

$\therefore$ g is 1-1.

$$\therefore f_Y(y) \overset{1\text{-}1}{=} f_X(x) \cdot \left| \frac{dx}{dy} \right|$$

$X = \frac{y-b}{a}$ since $a \neq 0$ $\quad\therefore \frac{dx}{dy} = \frac{1}{a}$.

$$\therefore \left| \frac{dx}{dy} \right| = \frac{1}{|a|}$$

$$\boxed{\therefore f_Y(y) = \frac{1}{|a|} \cdot f_X\left(\frac{y-b}{a}\right)}$$

Ex: Exponential $Y = e^X$ (1-1)

$\therefore X = \ln Y$ since $Y > 0$

$$\therefore \left| \frac{dx}{dy} \right| = \left| \frac{1}{y} \right| = \frac{1}{y} \quad \text{since } y > 0$$

$$\therefore f_Y(y) = \frac{1}{y} \cdot f_X(\ln y)$$

Say $X \sim N(\mu, \sigma_x^2)$: $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_x}\right)^2}$

$\therefore$ log-normal: $f_Y(y) = \frac{1}{y\sqrt{2\pi}\,\sigma_x} e^{-\frac{1}{2}\left(\frac{\ln y - \mu_x}{\sigma_x}\right)^2}$. $\left(Y \begin{smallmatrix} > 0 \\ = 0 \text{ else} \end{smallmatrix}\right)$

$$\therefore E_Y[Y] = e^{\mu_x + \sigma_x^2/2}$$

$$V_Y[Y] = e^{2\mu_x + \sigma_x^2}\left(e^{\sigma_x^2} - 1\right)$$

$\therefore Y \sim \text{LogNormal}(\mu_x, \sigma_x^2) \Longrightarrow \ln Y \sim N(\mu_x, \sigma_x^2)$

and $\ln Y^k \sim N(k\mu_x, k^2\sigma_x^2)$

(in class: $a=1$)

Ex: Square $Y = aX^2$ for $a>0$ ($\therefore$ not 1-1)

$y<0 \longrightarrow$ no real roots

$\therefore f(y) = 0$ if $y<0$

$y>0 \longrightarrow$ 2 solutions : $x = \pm\sqrt{\frac{y}{a}}$

$g(X) = Y = aX^2$.

$\therefore g'(x) = \frac{dy}{dx} = 2ax = \pm 2a\sqrt{\frac{y}{a}} = \pm 2\sqrt{ay}$

$\therefore |g'(x)| = 2\sqrt{ay}$

$\therefore f(y) = \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|}$ ($=0$ if $y<0$)

$= \frac{1}{2\sqrt{ay}}\left(f_X(\sqrt{y/a}) + f_X(-\sqrt{y/a})\right)$

Ex: $Z \sim N(0,1)$ $\therefore f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

Put $X = Z^2$

$\therefore f_X(x) = \frac{1}{2\sqrt{x}}\left(f_Z(\sqrt{x}) - f_Z(-\sqrt{x})\right) = \underbrace{\frac{1}{\cancel{2}\sqrt{x}}}_{x^{-\frac{1}{2}}} \cdot \cancel{2} f_Z(\sqrt{x})$

even: $f_Z(\sqrt{y})$

$= x^{-\frac{1}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{x})^2/2}$

$= \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x/2}$

$= \frac{x^{\frac{1}{2}-1} e^{-x/2}}{(2)^{1/2} \underbrace{\sqrt{\pi}}_{\Gamma(\frac{1}{2})}}$

$\therefore X \sim \gamma(\frac{1}{2}, 2)$ $\therefore X \sim \chi^2(1)$

$\gamma(\frac{1}{2}, 2)$

$r=1$

Ex: $X \sim \text{Cauchy}(0,1)$ $\qquad Y = X^2$

symmetric pdf, $\therefore f(x) = f(-x)$

$$f_Y(y) = \frac{1}{\sqrt{y}} \cdot f_X(\sqrt{y}) = \frac{1}{\sqrt{y}} \cdot \frac{1}{\pi(1+(\sqrt{y})^2)} = \frac{1}{\pi\sqrt{y}(1+y)}$$

Ex: $Y = X^2$, $\qquad E[Y] = ?$

method 1: $E_Y[Y] = \int_{-\infty}^{+\infty} y \cdot f_Y(y)\, dy = \int_{-\infty}^{+\infty} y \cdot \frac{1}{\pi\sqrt{y}(1+y)}\, dy$

method 2: $E_Y[Y] = E_X[g(X)] = E_X[X^2] = V[X] + (E[X])^2$

(optional)

Thm: random variable X w/ pdf, $f_X(x)$. Define $Y = g(X)$.

$$f_Y(y) = \sum_{k=1}^{n} f_X(x_k) \cdot \left| \frac{dx}{dy} \right|_{x = x_k}$$

roots of $Y = g(x)$

(from change of variable theorem)

Thm: random variables X and Y w/ joint pdf $f_{XY}(x,y)$.

Define $U = g_1(X,Y)$ $\qquad$ (Suppose g_1 and g_2 invertible)

$V = g_2(X,Y)$

$$\therefore f_{UV}(u,v) = f_{XY}\left(g_1^{-1}(u,v),\, g_2^{-1}(u,v)\right) \cdot \left| \underbrace{J(u,v)}_{\text{Jacobian}} \right| \qquad \frac{1}{|J(x,y)|}$$

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Sampling from a distribution

Ex: Want to draw iid $X_1, \dots, X_n \sim \exp(\theta)$

$$\left(X \sim \exp(\theta) : f(x) = \frac{1}{\theta} e^{-x/\theta} \right)$$

Trick: - draw r.s. from $U[0,1]$.

- back-solve for $X \sim f(x)$

If X has closed form F^{-1} ★ (inverse CDF)

Else: Use Accept-Reject (or other algorithm, e.g. Box-Muller for $N(0,1)$)

Review

g^{-1} exists iff g is 1-1 and ONTO

- $g: X \to Y$ is ONTO : $g(X) = Y$.

$$\forall y \in Y \ \exists x \in X : y = g(x)$$

- g is 1-1 iff

$$g(x_1) = g(x_2) \Longrightarrow x_1 = x_2 \quad \forall x_1, x_2.$$

$$\left(x_1 \neq x_2 \longrightarrow g(x_1) \neq g(x_2) \right)$$

Ex: $g(x) = e^x$. Say $g(x_1) = g(x_2)$

$\therefore e^{x_1} = e^{x_2}$ $\quad \therefore x_1 = \ln e^{x_1} = \ln e^{x_2} = x_2$ $\quad \therefore$ 1-1.

Continuous CDF w/ F^{-1})

Thrm: $Y = F(X) \sim U[0,1]$. (F^{-1} exists)

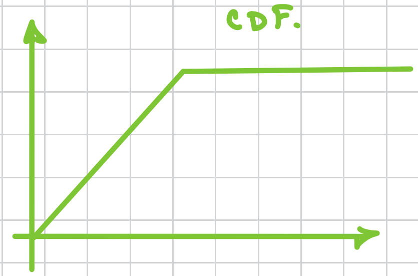
Prf: $F_Y(y) \overset{def}{=} P[Y \le y] = P[\{\omega \in \Omega : Y(\omega) \le y\}]$

$\overset{hyp}{=} P[F_X(X) \le y]$

$\overset{1-1}{=} P[X \le F_X^{-1}(y)]$

$\overset{\text{def CDF}}{=} F_X(F_X^{-1}(y))$

$\overset{1-1}{=} y$



$\therefore f_Y(y) = \frac{d}{dy} F_Y = \frac{dy}{dy} = 1.$

$\therefore Y \sim U(0,1)$ QED.

$\therefore$ turn it around.

Thrm: $U \sim U[0,1]$ and $X \sim F$ (continuous CDF) then $X = F^{-1}(U)$ has CDF F.

Prf: $F_U(u) = u \quad \forall u \in (0,1)$ since $U \sim U[0,1]$.

$\therefore P(X \le x) = P(F^{-1}(U) \le x)$

$= P(FF^{-1}(U) \le F(x))$

$= P(U \le F(x))$

$= F(x)$ QED.

Ex: Find $n=3$ r.s. (iid) draws from $\exp(2)$

Want: $X_1, X_2, X_3 \sim f(x) = \frac{1}{2} e^{-x/2}$ $(x>0)$

Sol: $Y = F(x)$ uniform from Theorem

$\therefore$ in general: $U = F(x) = 1 - e^{-x/\theta} \in (0,1)$ $(\exp(2))$

Backsolve: $\therefore X = -\theta \cdot \ln(1-U) \sim \exp(\theta)$

$u_k = 1 - e^{-x/2}$ $(\exp(2))$ $k = 1,2,3.$

Pick "random number" in $U[0,1]$ for u_k:

pick $u_1 = 0.10480 \Longrightarrow \therefore x_1 = 0.222$

realization

$u_2 = 0.24130 \Longrightarrow \therefore x_2 = 0.552$

$u_3 = 0.22368 \Longrightarrow \therefore x_3 = 0.506$

$\therefore$ iid draws: $\{0.222, 0.552, 0.506\} \sim \exp(2)$

Method: Draw iid $X_1, \dots, X_n$ from $X \sim f(x)$. If X continuous and F^{-1} exists

1. $U = F(X) \sim U[0,1]$
2. "Randomly" pick $u_k \in (0,1)$
3. Back-solve: $x_k = F^{-1}(u_k)$

Ex: Draw iid (standard Cauchy) $Z_1, \dots, Z_n \sim C(0,1)$

$$f(z) = \frac{1}{\pi} \cdot \frac{1}{1+z^2} \longleftarrow \text{continuous pdf.}$$

$$\therefore F(z) = \int_{-\infty}^{z} f(x)\,dx = \frac{1}{\pi} \tan^{-1} z + \frac{1}{2}.$$

$$\therefore u = F(z) \sim U(0,1)$$

$$u_k = \frac{1}{\pi} \tan^{-1} x_k + \frac{1}{2}$$

$$\therefore z_k = \tan\left(\pi\left(u_k - \frac{1}{2}\right)\right) \sim C(0,1)$$

SαS: $\alpha \in (0,2)$ $\qquad \alpha = 1 \longrightarrow$ Cauchy, $\alpha = 2 \longrightarrow$ Normal $\quad$ only closed forms.

C.F. $\phi(\omega) = e^{-\gamma|\omega|^{\alpha}}$

Q: what about $X_1, \dots, X_n \sim N(\mu, \sigma_x^2)$

Problem: no closed form F^{-1} $\quad \therefore$ use Box-Müller

Generating Normal realizations (Box-Müller, 1958)

Thrm: Pick iid $Y_1 \sim U(0,1)$ and $Y_2 \sim U(0,1)$ then X_1 and X_2 are iid $N(0,1)$ if

$$X_1 = \sqrt{-2 \cdot \ln Y_1} \cos(2\pi Y_2) \qquad X_2 = \sqrt{-2 \ln Y_1} \cdot \sin(2\pi Y_2)$$

(optional)

Pf The above transform $(X_1, X_2) = g(Y_1, Y_2)$ is 1-to-1

and maps the open unit-square $(0,1)^2$ ONTO $\mathbb{R}^2$

except for sets involving $X_1 = 0$ or $X_2 = 0$ (have zero-probability)

$\therefore$ Can use $f(x_1, x_2) \overset{1-1}{=} |J| \cdot f(y_1, y_2)$ with solutions:

$$Y_1 = \exp\left(-\frac{x_1^2 + x_2^2}{2}\right) \qquad Y_2 = \frac{1}{2\pi} \tan^{-1}\left(\frac{x_2}{x_1}\right)$$

with $X_1 \neq 0$ and $X_2 \neq 0$

$\therefore$ The Jacobian is

$$J = \det \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix}$$

$$= \det \begin{vmatrix} (-x_1)\exp\left(-\frac{x_1^2+x_2^2}{2}\right) & (-x_2)\exp\left(-\frac{x_1^2+x_2^2}{2}\right) \\ \frac{-x_2/x_1}{2\pi\left(1+x_2^2/x_1^2\right)} & \frac{1/x_1}{2\pi\left(1+x_2^2/x_1^2\right)} \end{vmatrix}$$

$$= \frac{-\left(1+x_2^2/x_1^2\right)\exp\left(-\frac{x_1^2+x_2^2}{2}\right)}{2\pi\left(1+x_2^2/x_1^2\right)} = -\frac{1}{2\pi}\exp\left(-\frac{x_1^2+x_2^2}{2}\right)$$

But iid $Y_k \sim U(0,1) \implies f(y_1, y_2) \overset{\text{ind}}{=} f(y_1) \cdot f(y_2) \overset{U(0,1)}{=} 1 \cdot 1 = 1$ on $(0,1)^2$

$\therefore f(x_1, x_2) \overset{1-1}{=} |J|\, f(y_1, y_2) \overset{Y_k \sim U(0,1)}{=} |J|$

$$= \frac{1}{2\pi}\exp\left(-\frac{x_1^2+x_2^2}{2}\right) = \left(\frac{1}{\sqrt{2\pi}} e^{-x_1^2/2}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-x_2^2/2}\right)$$

$= f(x_1) \cdot f(x_2)$ $\quad\therefore$ iid $X_k \sim N(0,1)$ QED.